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LETTER TO THE EDITORS

A DISCUSSION OF THE PAPER "ON THE SOLIDIFICATION OF A WARM LIQUID FLOWING OVER A COLD WALL"

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THE PROBLEM under discussion is that of predicting the transient growth of a frozen layer that forms when a warm liquid flows over a cold wall at constant temperature. Exact solutions for the instantaneous frozen layer thickness are rare and simple approximate solutions are difficult to derive. M. Elmas' work [1] is one attempt to develop an approximate solution. The objective of his paper is stated in his first paragraph, quoted as follows, "Recently there have been some attempts by several authors [1–4] to produce a closed yet simple relation giving the freezing rate of a warm liquid. Most of the solutions available are cumbersome and involve extensive computations. Below, we give a new analytical solution which is in a closed form and easy to use in practical situations." Included in his references [1–4] is our paper given in this discussion as [2].

Our objective in this discussion is to show that Elmas' solution, which is in the form of an integral equation that he states is to be solved by numerical or graphical techniques, can be solved analytically to yield a simple explicit algebraic expression. It will be shown that this expression has the same form as the second of four successive approximate solutions that were already given in [2].

There are two unknowns in this problem, the frozen layer thickness and the temperature distribution at any time during the growth period. Elmas' analysis and ours [2] are similar in that both were carried out using an integral technique. The transient one-dimensional heat conduction equation is placed in integral form and combined with the boundary conditions to yield an ordinary differential equation for the frozen thickness as a function of time. Elmas integrated this O.D.E. to obtain the following integral equation [Elmas' equation (18)].

$$\delta^2 = \frac{\int_0^{\Theta}\left[(T_w - 1) + (T_l - 1)\frac{h_l\delta}{k}\right]\mathrm{d}\Theta}{\int_0^1 \eta T\,\mathrm{d}\eta - \frac{1}{2}\left(T_l + \frac{L}{C_p t_f}\right)} \tag{1}$$

where δ = frozen layer thickness,
T = non-dimensional temperature, t/t_f,
Θ = time variable.

To evaluate the integral $\int_0^1 \eta T\,\mathrm{d}\eta$ Elmas assumed a constant temperature half way between the wall and freezing temperatures, and simplified equation (1) to give his final result equation (21) which is

$$\delta^2 = \frac{2\int_0^{\Theta}\left[T_w - 1 + (T_l - 1)\frac{h_l\delta}{k}\right]\mathrm{d}\Theta}{\frac{1}{2}(T_w + 1) - \left(T_l + \frac{L}{C_p t_f}\right)}. \tag{2}$$

This is the equation that Elmas states can be solved by numerical or graphical means. It is the objective of this discussion to show how equation (2) can be reduced to an algebraic expression, thereby, eliminating the need for numerical or graphical computations.

Differentiating (2) gives

$$2\delta\,\mathrm{d}\delta = \frac{2\left[T_w - 1 + (T_l - 1)\frac{h_l\delta}{k}\right]\mathrm{d}\Theta}{\frac{1}{2}(T_w + 1) - \left(T_l + \frac{L}{C_p t_f}\right)}.$$

Separate variables and integrate

$$\int_0^{\Theta}\mathrm{d}\Theta = \left[\tfrac{1}{2}(T_w + 1) - \left(T_l + \frac{L}{C_p t_f}\right)\right] \int_0^{\delta}\frac{\delta}{T_w - 1 + (T_l - 1)(h_l\delta/k)}\,\mathrm{d}\delta \tag{3}$$

which gives

$$\Theta = \left[\frac{T_w + 1}{2} - \left(T_l + \frac{L}{C_p t_f}\right)\right]\left[\frac{(T_w - 1)}{(T_l - 1)^2}\left(\frac{k}{h_l}\right)^2\right] \times\left[\left(\frac{T_l - 1}{T_w - 1}\right)\frac{h_l\delta}{k} - \ln\left|1 + \left(\frac{T_l - 1}{T_w - 1}\right)\frac{h_l\delta}{k}\right|\right]. \tag{4}$$

This solution is of the same form as the second of four successive approximate solutions reported as equation (13a) in [2] in which a linear profile between freezing temperature $T = 1$ and the wall temperature T_w was used to evaluate the integral in the denominator of equation (1).

REFERENCES

1. M. Elmas, On the solidification of a warm liquid flowing over a cold wall, *Int. J. Heat Mass Transfer* **13**, 1060–1062 (1970).
2. J. M. Savino and R. Siegel, An analytical solution for solidification of a moving warm liquid onto an isothermal cold wall, *Int. J. Heat Mass Transfer* **12**, 803–809 (1969).

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